Newton-Puiseux series for non-holonomic D-modules and factoring linear partial differential operators

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It is well-known that any polynomial equation $t(x,y)=0$ has $deg_y(t)$ (counting with multiplicities) zeroes being Newton-Puiseux series

$$y(x)=\sum_{i_0\le i<\infty} y_i x^{-i/q} \tag{1}$$

for suitable integers $q\ge 1, i_0$ and the coefficients y_i from an algebraically closed field.

In this paper an analogue of Newton-Puiseux series for linear partial differential equations $T=0$ is proposed. Whereas a Newton-Puiseux series is developed for a (plane) curve, we restrict ourselves with linear partial differential operators $T=\sum_{0\le j+l\le n} b_{j,l}d_x^j d_y^l$ in two derivatives d_x, d_y.

One of the principal features of Newton-Puiseux series is the appearance of fractional exponents. Thus, a question arises, what could be an analogue of fractional powers, so to say "fractional derivatives"? An evident observation shows that in the derivative $y'(x)=\sum_i(-i/q+1)y_{i-q}x^{-i/q}$ the i-th coefficient depends on the $(i-q)$-th coefficient of $y(x)$ itself.

That is why as a differential analogue of Newton-Puiseux series we suggest a *fractional-derivatives series* of the form

$$\sum_{0\le i<\infty} h_i G^{(-i/q)} \tag{2}$$

where h_i being elements of a differentially closed field F and $G^{(-i/q)}$ is called $(-i/q)$-*th fractional derivative* of G. The symbol $G=G^{(0)}=G_{(s_2,\dots,s_k)}(f_1,f_2,\dots,f_k)$ is defined by rational numbers $1>s_2>\cdots>s_k>0$ and $f_1,\dots,f_k\in F$ (if to continue the analogy with curves, G plays a role of a uniformizing element). For any rational s the s-th fractional derivative $G^{(s)}$ fulfills the identity

$$dG^{(s)}=(df_1)G^{(1+s)}+(df_2)G^{(s_2+s)}+\cdots+(df_k)G^{(s_k+s)}$$

where either a derivative $d=d_x$ or $d=d_y$. The common denominator q of $s_2,\dots,s_k$ plays a role similar to one of the common denominator of the exponents in a Newton-Puiseux series (1). The inequality $k\le q$ holds.

In a particular case $k=1$ we have $q=1$ and as G one can take $g(f_1)$ for any univariate ("undetermined") function g, provided that the composition makes sense, the fractional derivatives $G^{(s)}=g^{(s)}(f_1)$ for integers s. We note that finite sums

$$\sum_{i_0\le i\le i_1} h_i G^{(-i)}$$

(so, for $k = q = 1$) appear in the Laplace method as solutions of some second-order equations $T = 0$.

Theorem 0.1 *A linear partial differential equation $T = 0$ for operator T of order n has a Newton-Puiseux series solution of the form (2) with denominator $q \leq n$.*

As a consequence we design
an **algorithm** which finds all first-order $d_x + ad_y + c$ right divisors of T.

Corollary 0.2 *One can factor an operator T of order at most 3.*

A D-nodule $M \subset (F[d_x, d_y])^m$ is called *holonomic* if the degree of the Hilbert-Kolchin polynomial of M equals 0. The latter is equivalent to that the space of solutions of M is finite-dimensional over the subfield of constants of F. The next theorem generalizes the previous one.

Theorem 0.3 *A non-holonomic D-module M has a Newton-Puiseux series solution.*